Commutative Algebra | September 2024

Mid-Semester Exam | 100 Points

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) There are a total of **105** points in this paper. You will be awarded a maximum of **100** points.

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1. [10 + 10 + 10 = 30 Points] Let $f\colon A \to B$ be a map of rings. In each of the following cases, first prove the given inclusion of ideals. Next determine whether the two ideals are equal. If they are equal, give brief proofs. If they are unequal, give a counterexample involving suitable choices for A, B and the ideals involved.
   (i) $I^e J^e \subset (IJ)^e$ where I, J are ideals in A.
   (ii) $I^c J^c \subset (IJ)^c$ where I, J are ideals in B.
   (iii) $(\sqrt{I})^e \subset \sqrt{I^e}$ where I is an ideal in A.

2. [15 Points] Do any one out of (a) and (b).

   (a) Let $(A, \mathfrak{m}, k)$ be a local ring. Prove that $A[[x]]$ is also a local ring and identify its maximal ideal and its residue field.

   (b) Let $\{M_n\}_{n>0}$ be a set of modules over a ring A and let I be an ideal in A. Prove that $I \prod_n M_n \subset \prod_n IM_n$. If I is finitely generated, prove that the two modules are equal.

3. [15 Points] Let $\phi\colon A \hookrightarrow B$ be an injective map of nonzero rings. Prove that the image of the induced map of spectra $\phi^*\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is dense.
[*Hint: Intersect the image with any basic nonempty open subset of* Spec(A).]

4. [10 + 10 + 10 = 30 Points] Let R be an integral domain with fraction field $Q(R)$. We identify the localisations of R as subrings of $Q(R)$ via their natural embeddings.
   (i) Prove that
   $$R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}} = \bigcap_{\mathfrak{m}} R_{\mathfrak{m}}$$
   where $\mathfrak{p}$ ranges over all the prime ideals in R and $\mathfrak{m}$ over the maximal ones.
   [*Hint: For $x \in \bigcap_{\mathfrak{m}} R_{\mathfrak{m}}$, look at its denominator ideal, namely* $\{b \in R \mid bx \in R\}$.]
   (ii) Suppose q is a prime element in R. Prove that $R[1/q] = \bigcap_{q \notin \mathfrak{p}} R_{\mathfrak{p}}$.
   (iii) Now assume $R = F[x, y]$, a polynomial ring over a field F. Prove that
   $$\bigcap_{\mathfrak{m} \neq (x,y)} R_{\mathfrak{m}} = R[\frac{1}{x}] \cap R[\frac{1}{y}] = R.$$

5. [15 Points] Let $\phi\colon N \to M$ be a map of modules over a ring A with M finitely generated. Prove that if the natural induced map $N \otimes_A k(\mathfrak{p}) \to M \otimes_A k(\mathfrak{p})$ is onto for some prime $\mathfrak{p} \in \operatorname{Spec}(A)$, then there is an open neighbourhood U of $\mathfrak{p}$ such that the induced map $N \otimes_A k(\mathfrak{q}) \to M \otimes_A k(\mathfrak{q})$ is onto for all $\mathfrak{q} \in U$.